Round-Tour Voronoi Diagrams

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Outline

- Motivation
- Round-tour distances and round-tour Voronoi diagrams
- Basic properties
- Approximation algorithm
- Experiments on computation time
- Concluding remarks
Motivation
Voronoi diagram for the bookstores
Voronoi diagram for the bookstores
Voronoi diagrams for the restaurants
Voronoi diagrams for the restaurants
Definitions

$A$ : set of restaurants

$B$ : set of bookstores

For $a \in A$, $b \in B$ and any point $z \in \mathbb{R}^2$

Round - tour distance

$$l_z(a, b) = d(z, a) + d(a, b) + d(b, z)$$

Round - tour Voronoi region

$$V(A, B; a, b) = \{ z \in \mathbb{R}^2 \mid l_z(a, b) = \min_{a' \in A, b' \in B} l_z(a', b') \}$$
Round-tour Voronoi diagram for 3 restaurants and 3 bookstores
Definitions: order $k$ round-tour Voronoi diagram

$A_1, A_2, \ldots, A_k : k$ disjoint sets of generators

For $a_1 \in A, \ldots, a_k \in A_k$ and any point $z \in \mathbb{R}^2$

Order $k$ round-tour distance

$$l_z(a_1, \ldots, a_k) = \min \{d(z, a_{\tau(1)}) + \sum_{i=1}^{k-1} d(a_{\tau(i)}, a_{\tau(i+1)}) + d(a_{\tau(k)}, z)\}$$

Order $k$ round-tour Voronoi region

$$V(A_1, \ldots, A_k; a_1, \ldots, a_k)$$

$$= \{z \in \mathbb{R}^2 \mid l_z(a_1, \ldots, a_k) = \min_{a'_i \in A_i, i=1, \ldots, k} l_z(a'_1, \ldots, a'_k)\}$$
Related Work

• Ohyama and Suzuki 2000
  Consumers who visit several stores to search for goods
• Barequet et al. 2002
  Two-site Voronoi diagrams with various distances
• Ohyama 2004, 2005
  Consumers who visit stores until they find wanted goods
• Hanniel and Barequet 2009
  Complexity of triangle-perimeter two-site Voronoi diagram
Basic Properties
Property 1. Contours of the round-tour distance are ellipses.
Property 2. The boundary between $V(A, B; a, b)$ and $V(A, B; a, c)$ is one branch of the hyperbola with foci $c$ and $b$. 

\[ (a, c) \]

\[ c \]

\[ b \]

\[ a \]
Property 3. A Voronoi region of the round-tour Voronoi diagram is not necessarily connected.
Property 4. Let $a \in A, b \in B, z \in \mathbb{R}^2$ and $\varepsilon$ be a positive real number. For any $z' \in U(z, \varepsilon)$ the following inequality is satisfied.

$$l_z(a, b) \leq l_z(a, b) + 2\varepsilon$$
Property 4. Let $a \in A, b \in B, z \in \mathbb{R}^2$ and $\varepsilon$ be a positive real number. For any $z' \in U(z, \varepsilon)$ the following inequality is satisfied.

$$l_z'(a, b) \leq l_z(a, b) + 2\varepsilon$$
Property 5. For any $x, x' \in A$ and $y, y' \in B$, if 

$$d(x, y) > d(x', y) + d(x, y'),$$

then $V(A, B; x, y)$ is empty.
Property 6. If \( a \in V(B; b) \) or \( b \in V(A; a) \), then \( V(A, B; a, b) \) is nonempty.
Property 6. If $a \in V(B; b)$ or $b \in V(A; a)$, then $V(A, B; a, b)$ is nonempty.

This is because $a$ belongs to $V(A, B; a, b)$. 
Property 7. Let $|A| = m$ and $|B| = n$. Then, the number of nonempty Voronoi region can be as small as $\max(m, n)$. 
Property 8. Let $|A| = m$ and $|B| = n$. Then, the number of positive-area Voronoi region can be as small as 1.
Property 9. Let $|A| = m$ and $|B| = n$. Then, the number of Voronoi region can be as large as $mn$. 

![Diagram showing Voronoi regions with restaurants and bookstores marked.]
Digital Picture Approximation
Digital Picture Approximation

Naïve Method

For each pixel \( z \), find the pair \( a \in A \) and \( b \in B \) that minimizes \( l_z(a, b) \).
Digital Picture Approximation

Naïve Method

For each pixel \( z \), find the pair \( a \in A \) and \( b \in B \) that minimizes \( l_z(a, b) \).

Time complexity: \( O(n^2N^2) \)

\[ |A| = |B| = O(n) \text{ and } N \times N \text{ pixels} \]
Conditions for pruning \((a, b)\)

**Condition 1.** For any pixel \(z\)

\[
l_z(a, b) > l_z(a', b) \text{ for some } a' \in A, \text{ or } l_z(a, b) > l_z(a, b') \text{ for some } b' \in B.
\]

**Condition 2.** There exists \(a' \in A\) and \(b' \in B\)

such that

\[
d(a, b) > d(a', b) + d(a, b').
\]
Pruning of generator pairs

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<th>No. of all pairs</th>
<th>No. of pairs without regions</th>
<th>No. of pairs pruned by Condition 1</th>
<th>No. of pairs pruned by Condition 2</th>
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<td>2406 (100%)</td>
<td>2070 (86%)</td>
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<td>2399 (99.5%)</td>
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Almost tight but expensive

Fast
Digital Picture Approximation

Proposed Method
For each pixel $z$,

1. compute the length of the round tour under the assumption that $\mathcal{Z}$ belongs to the same Voronoi region as its neighbor.
2. Prune the generator $c$ with $d(z,c)$ greater than the length of the round tour.
3. Compute the true Voronoi region.
Computation Time

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<th>Computation Time</th>
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<td>Proposed method</td>
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Concluding Remarks

• We propose the round-tour Voronoi diagram, study mathematical properties, and construct a digital-picture approximation algorithm.

• Future work
  – Extension to three or more generator types
  – Applications to facility locations
  – Construction of an exact algorithm
Thank you very much.