Mathematics of Impossible Objects

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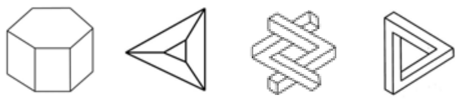
1. Picture-Interpretation Problem

Given a picture composed of line segments, judge whether it represents a polyhedral scene.



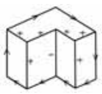
2. Mathematical Method for Interpretation

2.1 Assumptions

Assumption 1. Objects are solid bounded by planar faces.
Assumption 2. The view point is in general position.
Assumption 3. Each vertex is incident to exactly three faces.
Assumption 4. Visible edges only are drawn in the picture.
Assumption 5. The whole part of the object is drawn in the picture.

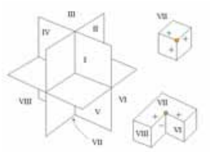
2.2 Construction of a Junction Dictionary

Classify the edges into three types: convex, concave, and silhouette, edges, which are represented by different labels.

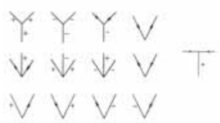


+ convex edge
− concave edge
→ silhouette edge

Enumerate all the possible combinations of labels around the junctions.

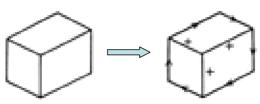


Thus obtain the list of possible junctions. This is called the "junction dictionary", because it can be used for interpretation of pictures.

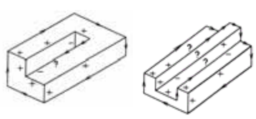


2.3 Picture Interpretation with the Junction Dictionary

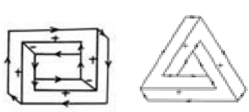
Assign labels to edges according to the junction dictionary, and thus obtain a candidate of interpretation.



The junction dictionary is not perfect because the obtained interpretation is not necessarily correct.

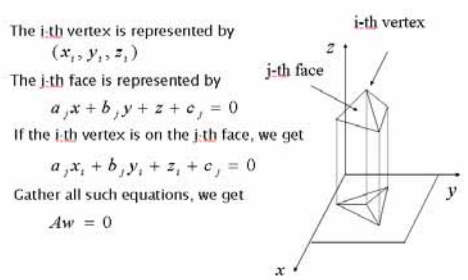


Judged "impossible" correctly

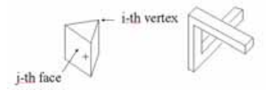


Incorrectly interpreted

2.4 Strict Judgment of the Correctness

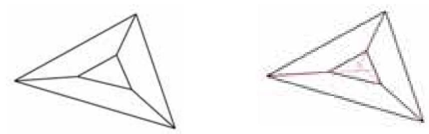


$$a_j x_i + b_j y_i + z_i + c_j > 0$$
$$Bw > 0$$



Theorem 1. The picture represents a polyhedral scene if and only if $Aw = 0$ and $Bw > 0$ has solutions.

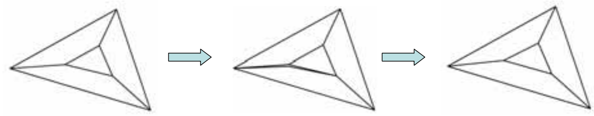
This theorem is too strict because the next picture is judged incorrect.



2.5 Robust and Flexible Judgment

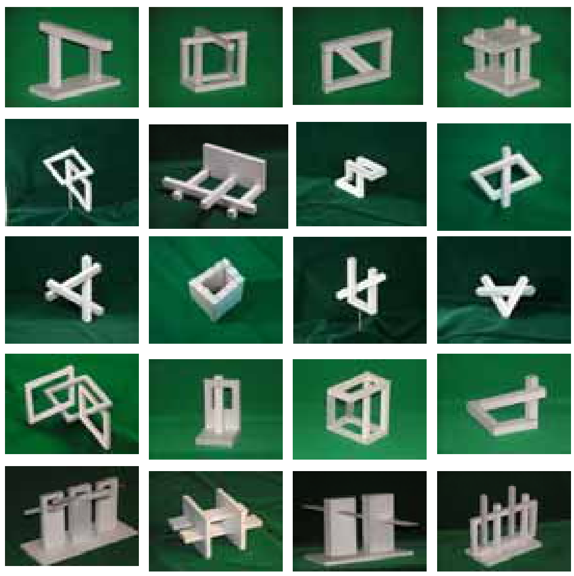
The overstrictness comes from redundancy of the system equations.

Theorem 2. The system of equations is non-redundant if and only if the following inequality is hold for any subset with $|F| \geq 2$:
$$|V| + 3|F| \geq |R| + 4,$$
where *V* represents the set of vertices, *F* the set of faces, and *R* the set of equations.



Input picture → Remove redundancy → Correct the picture

3. Realization of "Impossible Objects"



4. Invention of "Impossible Motions"